

On β -ensembles in RMT, by Rada Chhaibi

Tentative plan:

I/ The semi-circle law without matrices and without randomness

II/ Definition of β -ensembles and associated tridiagonal models

III/ Macroscopic consequences

① Semi-circle for $G\beta E$

② Bai-Yin thm

IV/ Microscopic consequences

① Stochastic Airy at the edge

② Brownian carousel

I/ The semi-circle law without matrices and without randomness

Informal Statement: Zeros of Hermite polynomials converge to semi-circle distribution.

1) Preliminaries on OPRL

OPRL = orthogonal polynomials on the real line

Let $\mu \in \mathcal{M}_+(\mathbb{R})$ with all moments (even exponential moments)

In $L^2(\mathbb{R}, \mu)$ with $\langle f, g \rangle_\mu = \int_{\mathbb{R}} f(x) \overline{g(x)} \mu(dx)$,

apply Gram-Schmidt to $(1, X, X^2, \dots)$ to get a sequence of monic OPRL $(P_0, P_1, P_2, \dots) =: \tilde{B}(\mu)$

$$\text{Let } p_n = \frac{P_n}{\|P_n\|_\mu^2}$$

Szegő's 3-term recurrence: there exists a sequence

$(a_n, b_n)_{n \geq 0}$ in $\mathbb{R} \times \mathbb{R}_+$ depending on μ such that

$$X P_n = P_{n+1} + a_n P_n + b_{n-1}^2 P_{n-1}$$

$$X p_n = b_n p_n + a_n p_n + b_{n-1} p_{n-1} \text{ with } b_n = \frac{\|P_{n+1}\|_\mu}{\|P_n\|_\mu}$$

Favard's theorem (1940s): informally, there is a correspondence

$$\mu \in \mathcal{M}_1(\mathbb{R}) \xrightarrow{\sim} (P_n)_{n \geq 0} \xrightarrow{\sim} (a_n, b_n)_{n \geq 0} \in (\mathbb{R} \times \mathbb{R}_+)^{\mathbb{N}}$$

2) OPRLs for the Gaussian: Hermite polynomials

$$\gamma(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Hermite polynomials $(He_n)_{n \geq 0}$ are the OPRLs associated to γ .

Basic facts given by Wikipedia:

$$\begin{cases} X He_n = He_{n+1} - n He_{n-1} \\ He_n' = n He_{n-1} \end{cases}$$

$$\leadsto a_j(\gamma) = 0 \text{ and } b_j(\gamma) = \sqrt{j+1}$$

3) The semi-circle for zeroes of H_{e_n}

Let $\mu_{sc}(dx) = \rho_{sc}(x) dx$ with $\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4-x^2} \mathbb{1}_{[-2,2]}(x)$

Fact 1: OPRLs have simple zeroes on $\mathbb{R}$ (consequence of the interlacing properties)

Fact 2: correct scaling is given by $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j^{(n)}/\sqrt{n}}$
so that $\left\{ \begin{array}{l} \int x \mu_n(dx) = 0 \\ \int x^2 \mu_n(dx) \xrightarrow{n \rightarrow \infty} 1 \end{array} \right. \Rightarrow \text{tightness of } (\mu_n)_n$

Theorem: $\mu_n \xrightarrow[n \rightarrow \infty]{\text{weakly}} \mu_{sc}$.

Proof: Let $s_n(z) = s_{\mu_n}(z) := \int_{\mathbb{R}} \frac{d\mu_n(x)}{z-x}$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

It is enough to prove that, for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s_n(z) \xrightarrow{n \rightarrow \infty} s_{sc}(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2}$$

Note that:

$$\begin{aligned} s_n(z) &= \frac{1}{n} \sum_{j=1}^n \frac{1}{z - \lambda_j^{(n)}/\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^n \frac{1}{z\sqrt{n} - \lambda_j^{(n)}} \\ &= \frac{1}{\sqrt{n}} \frac{d}{dx} \log H_{e_n}(x) \Big|_{x=\sqrt{n}z} \end{aligned}$$

$$\text{But } X H_{e_n} = H_{e_{n+1}} - n H_{e_{n-1}} \Leftrightarrow X = \frac{H_{e_{n+1}}}{H_{e_n}} - n \frac{H_{e_{n-1}}}{H_{e_n}}$$

$$\Leftrightarrow \frac{1}{n} \frac{He'_{n+1}}{He_{n+1}} = \left(X + \frac{He'_n}{He_n} \right)^{-1} \quad \text{where } He'_n = n He_{n-1}$$

$$\Leftrightarrow \frac{1}{n} \frac{d}{dX} \log He_{n+1} = \left(X + \frac{d}{dX} \log He_n \right)^{-1}$$

Setting $X = \sqrt{n} z$, this becomes $\sqrt{n+1} s_{n+1} \left(\sqrt{\frac{n}{n+1}} z \right) = \left(\sqrt{n} z + \sqrt{n} s_n(z) \right)^{-1}$

$$\text{that is } \frac{1}{z + s_n(z)} = \sqrt{\frac{n+1}{n}} s_{n+1} \left(\sqrt{\frac{n}{n+1}} z \right)$$

Fact 3: $\Lambda_j^{(n)}$ and $\Lambda_j^{(n+1)}$ interlace

This implies $s_{n+1} \left(\sqrt{\frac{n+1}{n}} z \right) = s_n(z) + o(1)$

$$\text{So } \frac{1}{z + s_n(z)} = s_n(z) + o(1) \Leftrightarrow s_n(z)^2 + z s_n(z) - 1 = o(1)$$

Then show $(s_n(z))_{n \geq 0}$ on $\{z \in \mathbb{C} : |\operatorname{Im} z| > \eta\}$ ($\eta > 0$ fixed)

is equicontinuous and bounded $\xRightarrow{\text{Ascoli}}$ precompact

And any accumulation point $s(z)$ satisfies

$$s(z)^2 + z s(z) - 1 = 0 \quad \Leftrightarrow \quad s(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2} \quad \square$$

II/ β -ensembles and associated tridiagonal models

1) Dyson's threefold way (1962)

$\beta = \dim_{\mathbb{K}} K$	1	2	4
Field K	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
group of orth. symmetries on K	O_n	U_n	Sp_n
Gaussian matrix ensembles	GOE	GUE	GSE

Defining GOE, GUE, GSE:

Let $\mathcal{N}_K$ be a Gaussian on K (at every coordinate)

such that $E[\mathcal{N}_K] = 0$ and $E[|\mathcal{N}_K|^2] = E[\mathcal{N}_K \mathcal{N}_K^*] = 1$

Then consider $G_n^{(\beta)} = (\mathcal{N}_{K,i;j})_{1 \leq i,j \leq n} \in \mathcal{M}_n(K)$

where the $(\mathcal{N}_{K,i;j})_{1 \leq i,j \leq n}$ are iid.

$$\text{and set } G\beta E_n = \frac{G_n^{(\beta)} + (G_n^{(\beta)})^*}{\sqrt{2\beta}}$$

Let $\Lambda^{(n,\beta)} = (\Lambda_1^{(n,\beta)}, \dots, \Lambda_n^{(n,\beta)}) \in \mathbb{R}^n$ be the eigenvalues of $G\beta E_n$.

Proposition: For all $\beta \in \{1, 2, 4\}$, $\Lambda^{(n)}$ has distribution

$$\frac{1}{Z_n^{(\beta)}} |\Delta(\Lambda)|^\beta \exp\left(-\frac{\beta}{2} \|\Lambda\|^2\right) d\Lambda \rightarrow \text{Lebesgue on } \mathbb{R}^n$$

where $\Delta(\Lambda) = \prod_{i < j} (\Lambda_i - \Lambda_j)$.

Generalizations:

- ① $\beta = 1, 2, 4$ becomes any $\beta > 0 \leadsto G\beta E_n$ includes the scaling $\Lambda \rightarrow \frac{\Lambda}{\sqrt{n}}$
- ② Replacing $\exp(-\frac{\beta}{2} \|\Lambda\|^2) = \exp(-\frac{\beta_n}{2} \sum_{j=1}^n \frac{\Lambda_j^2}{n})$ by $\exp(-\frac{\beta_n}{2} \sum_{j=1}^n V(\Lambda_j))$
for some $V: \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ growing fast enough at $\pm\infty$.

⚠ A priori no matrices!

Definition: General β -ensemble with potential V ($V\beta E_n$?)

is the r.v. $\Lambda^{(n)} \in \mathbb{R}^n$ with distribution

$$\frac{1}{Z_n^{(\beta, V)}} |\Delta(\Lambda)|^\beta \exp\left(-\frac{\beta_n}{2} \sum_{j=1}^n V(\Lambda_j)\right) d\Lambda$$

Remark: Physical interpretation: Gibbs measure $\frac{1}{Z_n} e^{-\beta \mathcal{E}_n(\Lambda)} d\Lambda$
where Z_n = partition function

$$\beta = 1/k_B T \quad \beta \rightarrow \infty \Leftrightarrow T \rightarrow 0 \text{ "crystallization"}$$

$\mathcal{E}_n(\Lambda)$ = energy / Hamiltonian of config. Λ (up to n^2)

$$\text{Here } \mathcal{E}(\Lambda) = \underbrace{\frac{1}{n} \sum_{j=1}^n V(\Lambda_j)}_{\text{confining potential}} - \underbrace{\frac{1}{n^2} \sum_{i < j} \log |\Lambda_i - \Lambda_j|}_{\text{logarithmic repulsion}}$$

see Hiai-Petz
book

logarithmic repulsion: pairwise
interaction for Coulomb electrostatic
force in dimension 2.

$\leadsto$ hence the name "Coulomb gas" in dim 2.

In this class: focus on $G\beta E_n$

2) Tridiagonal matrix models

following Dumitriu-Edelman (2002)

Question: Is there a simple random matrix $J_n^{(\beta)}$ whose spectrum is $G\beta E_n$ for any $\beta > 0$?

Theorem (DE 2002): Consider the symmetric tridiagonal matrices

$$\tilde{J}_n^{(\beta)} := \sqrt{\frac{2}{\beta}} \begin{pmatrix} \mathcal{N}_n & \chi_{\beta(n-1)/2} & & (0) \\ \chi_{\beta(n-1)/2} & \mathcal{N}_{n-1} & \chi_{\beta(n-2)/2} & \\ & \chi_{\beta(n-2)/2} & \mathcal{N}_{n-2} & \ddots \\ (0) & & \ddots & \chi_{\beta/2} & \mathcal{N}_1 \end{pmatrix} \quad \text{the one introduced by D-E}$$

$$J_n^{(\beta)} := \sqrt{\frac{2}{\beta}} \begin{pmatrix} \mathcal{N}_1 & \chi_{\beta/2} & & (0) \\ \chi_{\beta/2} & \ddots & \ddots & \\ & \ddots & \mathcal{N}_{n-2} & \chi_{\beta(n-2)/2} \\ (0) & \chi_{\beta(n-2)/2} & \mathcal{N}_{n-1} & \chi_{\beta(n-1)/2} \\ & \chi_{\beta(n-1)/2} & \mathcal{N}_n & \end{pmatrix}$$

Then $\text{Spec } J_n^{(\beta)} = \text{Spec } \tilde{J}_n^{(\beta)}$ has distribution $G\beta E_n$.

Link with Hermite polynomials:

• $J_n^{(\beta)} = \begin{pmatrix} a_0 & b_0 & & (0) \\ b_0 & a_1 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ (0) & & b_{n-1} & a_n \end{pmatrix}$ is a Jacobi matrix

Fact: its characteristic polynomial $P_n(x) = \det(xI_n - J_n)$

schisties $X P_n = P_{n+1} + a_n P_n + b_{n-1}^2 P_{n-1}$

• $J_n^{(\beta=\infty)}$ gives $\begin{cases} a_j = 0 \\ b_j = \sqrt{j+1} \end{cases}$ so $P_n^{(\beta=\infty)} = He_n$.

and $\Lambda^{(n, \beta=\infty)}$ is the zeroes of He_n .

This means zeroes of He_n are the minimizers of

$$E_n(\Lambda) = \frac{\|\Lambda\|^2}{2} - \sum_{i < j} \log |\Lambda_i - \Lambda_j|$$

3) Relationship to OPRL

Consider the linear (self-adjoint) morphism $X: L^2(\mu) \rightarrow L^2(\mu)$
 $f \mapsto Xf$

$$\langle Xf, g \rangle_\mu = \int x f(x) \overline{g(x)} d\mu(x) = \langle f, Xg \rangle_\mu$$

Recall (p_n) is an orthonormal basis with 3-term recurrence

$$X p_n = b_n p_{n+1} + a_n p_n + b_{n-1} p_{n-1}$$

So the matrix of $f \mapsto Xf$ in the basis $(p_0, p_1, \dots)$

is the Jacobi matrix $J(a, b) = \begin{pmatrix} a_0 & b_1 & & (0) \\ b_1 & a_1 & & \\ & \ddots & \ddots & b_{n-1} \\ (0) & & b_{n-1} & a_n & \ddots \end{pmatrix}$

We introduce the truncations:

$$J_n(a, b) = \begin{pmatrix} a_0 & b_1 & & (0) \\ b_1 & a_1 & & \\ & \ddots & \ddots & b_{n-1} \\ (0) & & b_{n-1} & a_n \end{pmatrix} \quad \tilde{J}_n(a, b) = \begin{pmatrix} a_n & b_{n-1} & & (0) \\ a_n & a_{n-1} & & \\ & \ddots & \ddots & b_0 \\ (0) & & b_0 & a_0 \end{pmatrix}$$

Proposition: Let $p \in \mathbb{Z}_+$ and the associated $(P_j)_{j \geq 0}$
and $(a_j, b_j)_{j \geq 0}$.

Then $P_n(x) = \det(XI_n - J_n(a, b)) = \det(XI_n - \tilde{J}_n(a, b))$

Proof: Same for J_n and $\tilde{J}_n$.

Let $Q_n = \det(XI_n - J_n)$

$$Q_{n+1} = \begin{vmatrix} XI_n - J_n & \begin{smallmatrix} 0 \\ \vdots \\ 0 \\ b_{n-1} \end{smallmatrix} \\ \hline 0 \dots 0 & b_{n-1} & X - a_n \end{vmatrix} = (X - a_n) Q_n - (-b_{n-1}) \begin{vmatrix} XI_{n-1} - J_{n-1} & 0 \\ \hline 0 & b_{n-1} \end{vmatrix}$$

$$= (X - a_n) Q_n - b_{n-1}^2 Q_{n-1}$$

So $XQ_n = Q_{n+1} + a_n Q_n + b_{n-1}^2 Q_{n-1}$ Szegő 3-term recurrence

so $Q_n = P_n$. □

Interpretation of $G^\beta E_n$ and DE's model

$$\Lambda^{(n, \beta)} = \text{Spec } J_n^{(\beta)} \rightarrow \text{matrix model of DE}$$

$$= \text{Zeros of } \det(XI_n - J_n^{(\beta)})$$

$$= P_n^{(\beta)} \text{ } n^{\text{th}} \text{ term in an OPRL recurrence.}$$

$$J_n^{(\beta)} \xrightarrow{\beta \rightarrow \infty} J_n^{(\beta=\infty)} = J_n(a, b) \text{ with } a_j = 0, b_j = \sqrt{j+1}$$

So $P_n^{(\beta=\infty)}$ satisfies $XP_n^{(\beta=\infty)} = P_{n+1}^{(\beta=\infty)} - n P_{n-1}^{(\beta=\infty)}$

so $P_n^{(\beta=\infty)} = H_n$

Non trivial consequence:

Zeros of $H_{\alpha} = \Lambda^{(n, \beta=\infty)} = \underset{\Lambda \in \mathbb{R}^n}{\text{Argmin}} \left(\frac{\|\Lambda\|^2}{2} - \sum_{i < j} \log |\Lambda_i - \Lambda_j| \right)$
see Saff, Totik - Theory of logarithmic potential

4) How to guess DE's model?

The proof is a relatively simple calculation, so we rather explain the intuition for $\beta=2$.

Intuition 1: Numerical linear algebra

How does a computer diagonalize a full Hermitian matrix?

$$\text{Write } \Pi_n = \left(\begin{array}{c|c} a & x^* \\ \hline x & \Pi_{n-1} \end{array} \right)$$

$$\text{Then } \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array} \right) \Pi_n \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^* \end{array} \right) = \left(\begin{array}{c|c} a & (Qx)^* \\ \hline Qx & Q\Pi_{n-1}Q^* \end{array} \right) \text{ for } Q \in U_{n-1}.$$

We can take Q s.t. $Qx = \|x\|e_1$, then

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array} \right) \Pi_n \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^* \end{array} \right) = \left(\begin{array}{c|c} a & \|x\| 0 \dots 0 \\ \hline 0 & \tilde{\Pi}_{n-1} \end{array} \right)$$

Repeat for $\tilde{\Pi}_{n-1}$: in the end we get a tridiagonal matrix.

Intuition 2: Gaussians should be tractable through this reduction (Trotter's reduction).

$$\begin{aligned}
 \text{Spec GUE}_n &= \text{Spec} \left(\frac{N_n \mid x^*}{x \mid \text{GUE}_{n-1}} \right) \\
 &= \text{Spec} \left(\frac{N_n \mid (Qx)^*}{Qx \mid Q \text{GUE}_{n-1} Q^*} \right) \\
 &\stackrel{\text{law}}{=} \text{Spec} \left(\frac{N_n \mid \underbrace{\chi_{n-1}}_{\|x\| \text{ is } \chi_{n-1} \text{ distributed}} \mid 0-0}{\begin{smallmatrix} \chi_{n-1} \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \mid \text{GUE}_{n-1}} \right) \\
 \text{repeat} \quad &= \text{Spec}(\tilde{J}_n^{(\beta=2)})
 \end{aligned}$$

Likewise for $\beta=1$ or $\beta=4$.

More intuition / Philosophy / Musing: Edelman "Ghosts and Shadows"

formulas inspired by the idea $\beta = \dim_{\mathbb{R}} K_{\beta}$ ($K_1 = \mathbb{R}, K_2 = \mathbb{C}, K_4 = \mathbb{H}$)

O_{β} = group of ON symmetries on K^n for $\beta = 1, 2, 4$

" = " Hypergroup of Jack polynomials for general $\beta > 0$
 generalization of groups $\rightarrow J_2^{(\alpha)}$ with $\alpha = 2/\beta$

III/Intermezzo: more OPRLs

What happens to $J(\mu) = J(a, b)$ if μ is atomic?

Let $\mu = \sum_{j=1}^n w_j \delta_{x_j}$. Then $\dim L^2(\mu) = n$.

So Gram-Schmidt stops: $(P_0, P_1, \dots, P_{n-1})$ is an orthogonal basis and then $P_n = 0$ in $L^2(\mu)$.

But P_n is monic of degree n so $P_n = \prod_{j=1}^n (x - x_j)$.

We have $b_{n-1} = 0$ and the rest is undefined:

$$J(\mu) = \left(\begin{array}{cccc|c} a_0 & b_0 & & & \\ b_0 & a_1 & & & \\ & & \ddots & & \\ & & & b_{n-2} & \\ \hline & & & b_{n-2} & a_{n-1} & 0 \\ & & & & & 0 \end{array} \right) = \left(\begin{array}{c|c} J_n(\mu) & 0 \\ \hline 0 & * \end{array} \right)$$

Numerical quadrature: (called "mechanical quad." by Szegő)

it is the idea of finding points $(x_j^{(n)})_{1 \leq j \leq n}$ and

weights $(w_j^{(n)})_{1 \leq j \leq n}$ s.t. $\int_{\mathbb{R}} f d\mu \stackrel{!}{=} \sum_{j=1}^n w_j^{(n)} f(x_j^{(n)})$

equality for certain f 's?

Theorem: Let $\mu \in \mathcal{P}_n(\mathbb{R})$ non atomic (so that $J(\mu)$ is infinite)

Let μ_n be the measure whose Jacobi matrix is

$$J(\mu_n) = \left(\begin{array}{c|c} J_n(\mu) & 0 \\ \hline 0 & * \end{array} \right). \text{ Then:}$$

• μ_n is atomic with $\mu_n = \sum_{j=1}^n w_j \delta_{x_j}$

- $P_n = \prod_{j=1}^n (x - I_j)$

- $\forall f \in C_{2n+1}[X], \int_{\mathbb{R}} f d\mu = \sum_{j=1}^n w_j^{(n)} f(I_j^{(n)})$

Examples:

- Gauss-Lobatto for $\gamma(dx) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$

- Gauss-Legendre for $\mathbb{1}_{[0,1]}(x) dx$

- Gauss-Laguerre for $e^{-x} \mathbb{1}_{x>0} dx$

→ integration using zeroes of OPRL as quadrature pts.

Puzzling remark: $J^{(\beta)} = \sqrt{\frac{2}{\beta}} \begin{pmatrix} \mathcal{N}_1 & \chi_{\beta/2} & (0) \\ \chi_{\beta/2} & \mathcal{N}_2 & \ddots \\ (0) & \ddots & \ddots \end{pmatrix}$ infinite

Then, by Faraut's theorem, $J^{(\beta)} = J(\mu_\beta)$ for some μ_β :

- μ_β is random

- $\mu_{\beta=\infty} = \gamma(dx)$

- μ_β = fractal random measure related to GTC.
(proven in the circular setting by Chhaibi-Najnudel)

IV/Macroscopic consequences of the tridiagonal model

1) Controlling $J_n^{(\beta)} - J_n^{(\beta=\infty)}$

Recall $J_n^{(\beta)} = \sqrt{\frac{2}{\beta}} J_n(a, b)$ with
$$\begin{cases} a_j = \mathcal{N}_j \\ b_j = \chi_{\beta(j+1)/2} \end{cases}$$

So $J_n^{(\beta)} - J_n^{(\beta=\infty)} = \sqrt{\frac{2}{\beta}} J_n(A, B)$ with
$$\begin{cases} A_j = \mathcal{N}_j \\ B_j = \chi_{\beta(j+1)/2} - \sqrt{\frac{\beta}{2}(j+1)} \end{cases}$$

Then $B_j \xrightarrow{j \rightarrow \infty} \sqrt{\frac{2}{\beta}} + \frac{1}{2} \mathcal{N}(0, 1)$

Thus $\|J_n^{(\beta)} - J_n^{(\beta=\infty)}\|_2^2 = \frac{2}{\beta} \left(\sum_{i=0}^{n-1} A_i^2 + 2 \sum_{j=0}^{n-2} B_j^2 \right)$

$\sim_{n \rightarrow \infty} \frac{2}{\beta} \left(n + 2n \cdot \frac{1}{4} \right) \sim \frac{3n}{\beta}$

$\|J_n^{(\beta)} - J_n^{(\beta=\infty)}\|_{op} \leq \frac{2}{\beta} \left(\underbrace{\|\text{diag } \mathcal{N}_i\|_{op}}_{\substack{\max_{1 \leq i \leq n} |\mathcal{N}_i| \sim \sqrt{2 \log n}} + 2 \underbrace{\left\| \begin{pmatrix} \mathcal{B}_0 & & \\ & \ddots & \\ & & \mathcal{B}_{n-2} \end{pmatrix} \right\|_{op}}_{= O(n^{\frac{1}{2}}) \text{ similarly}} \right)$

$= O(n^{\frac{1}{2}}) \quad \forall \varepsilon > 0.$

2) Semi-circle for $G\beta E_n$

$\Lambda^{(n, \beta)} = \text{Spec}(J_n^{(\beta)}) \stackrel{(d)}{=} G\beta E_n$. Set $\mu_n^{(\beta)} = \frac{1}{n} \sum_{j=1}^n \delta_{\Lambda_j^{(n)}/\sqrt{n}}$.

Theorem: $\forall \beta > 0, \quad \mu_n^{(\beta)} \xrightarrow[n \rightarrow \infty]{w} \mu_{sc}$

Proof 1: $s_n^{(\beta)}(z) = s_{\mu_n^{(\beta)}}(z) = \frac{1}{n} \sum_{j=1}^n \frac{1}{z - \lambda_j^{(\beta)} / \sqrt{n}} = \frac{1}{n} \text{Tr} \left[\underbrace{\left(z - \mathcal{J}_n^{(\beta)} \right)^{-1}}_{\text{resolvent}} \right]$

Resolvent formula: $(z - A)^{-1} - (z - B)^{-1} = (z - A)^{-1} (A - B) (z - B)^{-1}$

with $A = \mathcal{J}_n^{(\beta)} / \sqrt{n}$ and $B = \mathcal{J}_n^{(\beta=\infty)} / \sqrt{n}$

$$\Rightarrow |s_n^{(\beta)}(z) - s_n^{(\beta=\infty)}(z)| = \frac{1}{n} \left| \text{Tr} \left(\left(z - \frac{\mathcal{J}_n^{(\beta)}}{\sqrt{n}} \right)^{-1} \left(\frac{\mathcal{J}_n^{(\beta)} - \mathcal{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right) \left(z - \frac{\mathcal{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right)^{-1} \right) \right|$$

Hölder inequality for

Schatten norms

$$\left(\leq \frac{1}{n} \left\| \frac{\mathcal{J}_n^{(\beta)} - \mathcal{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right\|_1 \times \left\| \left(z - \frac{\mathcal{J}_n^{(\beta)}}{\sqrt{n}} \right)^{-1} \right\|_{\text{op}} \times \left\| \left(z - \frac{\mathcal{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right)^{-1} \right\| \right)$$

$$|\text{Tr}(XY)| \leq \|X\|_p \|Y\|_q$$

with $\|X\|_p = (\sum \sigma_i(X)^p)^{1/p}$, $\sigma_i(X)$ singular values of X

Crude bounds: for any A , $\|(z - A)^{-1}\|_{\text{op}} = \sup_{\lambda \in \text{Spec } A} \frac{1}{|z - \lambda|} \leq \frac{1}{|\text{Im } z|}$
 for any A, B , $\|A - B\|_1 \leq \sqrt{n} \|A - B\|_2$

$$\text{So } |s_n^{(\beta)}(z) - s_n^{(\beta=\infty)}(z)| \leq \frac{1}{\sqrt{n}} \left\| \frac{\mathcal{J}_n^{(\beta)} - \mathcal{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right\|_2 \frac{1}{(\text{Im } z)^2} = O\left(\frac{1}{\sqrt{n} (\text{Im } z)^2}\right)$$

$$\xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$



Proof 2: Same OT & Wieland-Hoffmann.

$$\text{Recall } W_2(\mu, \nu) = \inf_{\substack{\mathcal{L}(X) = \mu \\ \mathcal{L}(Y) = \nu}} E[\|X - Y\|^2]^{1/2}$$

$$= \inf_{\pi \in \Pi(\mu, \nu)} \left(\int \|x - y\|^2 \pi(dx, dy) \right)^{1/2}$$

measures de marginales μ et ν .

Here $\mu = \frac{1}{n} \sum \delta_{x_i}$, $\nu = \frac{1}{n} \sum \delta_{y_j}$.

Then, a coupling π can be written $(\pi_{ij})_{1 \leq i, j \leq n} = n \tilde{\pi}$
with $\tilde{\pi}$ bi-stochastic, $\pi_{ij} = \pi(\{x_i\} \times \{y_j\})$

$$\text{so } W_2(\mu, \nu) = \inf_{\tilde{\pi} \in \mathcal{B}_n} \left(\frac{1}{n} \sum_{i,j} \|x_i - y_j\|^2 \tilde{\pi}_{ij} \right)^{1/2}$$

bistochastic

Fact (Choquet): $W_2(\mu, \nu) = \inf_{\sigma \in \mathcal{S}_n} \left(\frac{1}{n} \sum_{i=1}^n \|x_i - y_{\sigma(i)}\|^2 \right)^{1/2}$

Here, we want

$$\sqrt{n} W_2(\mu_n^{(\beta)}, \mu_n^{(\beta=\infty)}) = \inf_{\sigma \in \mathcal{S}_n} \left(\sum_{i=1}^n \left\| \underbrace{\frac{\Lambda_i^{(n,\beta)}}{\sqrt{n}}}_{\lambda_i(A)} - \underbrace{\frac{\Lambda_{\sigma(i)}^{(n,\beta=\infty)}}{\sqrt{n}}}_{\lambda_i(B)} \right\|^2 \right)$$

Wieland-Hoffman

$$\leq \|A - B\|_2 = \frac{1}{\sqrt{n}} \|\mathcal{J}_n^{(\beta)} - \mathcal{J}_n^{(\beta=\infty)}\|_2$$

$$\text{So } W_2(\mu_n^{(\beta)}, \mu_n^{(\beta=\infty)}) \leq \frac{1}{n} \|\mathcal{J}_n^{(\beta)} - \mathcal{J}_n^{(\beta=\infty)}\|_2 = O\left(\frac{1}{\sqrt{n}}\right) \quad \square$$

2) Close to the edge: Bai-Yin

Theorem: $\frac{\Lambda_1^{(n,\beta)}}{\sqrt{n}} = \frac{\|\mathcal{J}_n^{(\beta)}\|}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 2$

Proof: • Lower bound obvious from CV to semi-circle:

$$\forall \varepsilon > 0, \liminf_{n \rightarrow \infty} \mu_n((2-\varepsilon, \infty)) \underset{\text{Portmanteau}}{\geq} \mu_{sc}((2-\varepsilon, \infty)) > 0.$$

• Upper bound

First idea: $(2+\varepsilon) I_n - \frac{\tilde{J}_n^{(\beta)}}{\sqrt{n}} \geq 0$?

$$\Leftrightarrow \forall X, \langle X, (2+\varepsilon) X - \frac{\tilde{J}_n^{(\beta)}}{\sqrt{n}} X \rangle \geq 0$$

Second idea: $2I_n - \frac{\tilde{J}_n^{(\beta)}}{\sqrt{n}} \approx \begin{pmatrix} 2 & \frac{-X_{\beta(n-1)/2}}{\sqrt{\beta-2}} & & \\ & 2 & \ddots & \\ & & \ddots & \\ & & & 2 \end{pmatrix} \approx \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix} = -\Delta_{[1,n]}$

and $-\Delta$ is positive!

$$(-\Delta f)(k) = 2f(k) - f(k+1) - f(k-1)$$

with $f(0) = 0 = f(n+1)$

proof: $\langle f, (-\Delta) f \rangle = 2 \sum_{i=1}^n f(i)^2 - \sum_{(i,j): |i-j|=1} f(i) f(j)$

$$= \frac{1}{2} \sum_{(i,j): |i-j|=1} (f(i) - f(j))^2 \geq 0$$

Actual proof of the upper bound:

$$\bullet \left| \frac{\Lambda_1^{(n,\beta)}}{\sqrt{n}} - \frac{\Lambda_1^{(n,\beta=\infty)}}{\sqrt{n}} \right| \leq \left\| \frac{\tilde{J}_n^{(\beta)} - \tilde{J}_n^{(\beta=\infty)}}{\sqrt{n}} \right\|_{op} = \frac{O(n^\varepsilon)}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0$$

so we only need to deal with the case $\beta = \infty$.

$$\bullet \langle X, ((2+\varepsilon) I_n - \frac{\tilde{J}_n^{(\beta=\infty)}}{\sqrt{n}}) X \rangle = 2 \sum X_i^2 - \sum_{(i,j): |i-j|=1} X_i X_j \sqrt{1 - \frac{i+j}{n}}$$

$$= \dots = \sum_{(i,j): |i-j|=1} \frac{1}{2} \sqrt{1 - \frac{i+j}{n}} (X_i - X_j)^2 + \sum_{i=1}^n \left(2 - \sqrt{1 - \frac{i-1}{n}} \sqrt{1 - \frac{i}{n}} \right) X_i^2$$

$$\geq 0$$



IV/ Microscopic consequences

1) Stochastic Airy Operator (SAO)

Following Krishnapur - Rider - Virag

Idea: $2I_n - \frac{J_n^{(\beta)}}{\sqrt{n}} \simeq -\Delta_{[1,n]} + \text{stuff at smaller scale}$

$$c n^{2/3} \left(2I_n - \frac{J_n^{(\beta)}}{\sqrt{n}} \right) \simeq -\Delta_{L^2(\mathbb{R}_+)} + x + \frac{2}{\sqrt{\beta}} W'_x \\ =: \text{SAO}_\beta \text{ (Dirichlet boundary)}$$

Theorem: $c n^{2/3} \left(2 - \frac{\Lambda_k^{(n,\beta)}}{\sqrt{n}} \right) \xrightarrow{n \rightarrow \infty} \Lambda_k(\text{SAO}_\beta)$
 $\Lambda_k^{(n,\beta)}$ $\swarrow$
 k -th largest eigenvalue of $G\beta E_n$

In particular (Courant - Fisher):

$$c n^{2/3} \left(2 - \frac{\Lambda_1^{(n,\beta)}}{\sqrt{n}} \right) \longrightarrow \Lambda_1(\text{SAO}_\beta) = \inf_{\|f\|_{L^2(\mathbb{R}_+)}^2 = 1} \langle f, \text{SAO}_\beta f \rangle \\ = \inf_{\|f\|_{L^2(\mathbb{R}_+)}^2 = 1} \left(\int_0^\infty f'(x)^2 dx + \int_0^\infty x f(x)^2 dx + \sqrt{\frac{2}{\beta}} \int_0^\infty f(x)^2 dW_x \right) \\ \stackrel{(d)}{=} \text{TW}_\beta \text{ (non trivial!)}$$