Towards a computational theory of star-shaped bodies.

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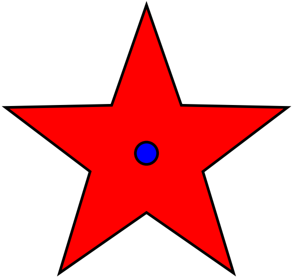
- *A* **star-body** *is a body which is star shaped with respect to the origin.*



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4. How to model the **space** of star-bodies?

Solving questions (2) *and* (3) *would give us a* **computational theory of star-bodies**.

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Exercise. Prove that for every star-body B we have $\rho_B(u)\gamma_B(u) = 1$.

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Theorem.

The star-source set of B contains a ball of radius $r > 0$ around the origin if and only if the gauge function $\gamma_B : S^2 \to \mathbb{R}$ is Lipschitz continuous. In this case $1/r$ is a valid Lipschitz constant.

We will focus on **Lipschitz star bodies**.

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- *The ring of continuous real-valued functions on the sphere $C(S)$ has the following two properties:*
  - *It is a Banach space with the supremum norm*

$$\|f-g\|_\infty := \sup_{u \in S^{n-1}} |f(u)-g(u)|$$

  - *It contains an explicit family, the restrictions of polynomials in $\mathbb{R}[x_1,\ldots,x_n]$ to S^{n-1}.*

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  - *It contains an explicit family, the restrictions of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ to S^{n-1}.This algebra,denoted $\mathbb{R}[S]$ is* **dense** *in $C(S)$ by the Stone-Weierstrass Theorem.*

Polystar approximations

It follows that the following families of bodies are universal approximators

Definition.

A star body is called **polyradial** *(resp.* **polygauge***) if its radial function (resp. gauge function)* $f : S^{n-1} \to \mathbb{R}$ *is the restriction of a multivariate polynomial to* S^{n-1}*.*

Polystar approximations

Consider the polynomial $p(x, y) = 32x^6 + 32y + 128$ *for* $(x, y) \in S$. *This defines a polygauge body* L_1 *via* $\gamma_{L_1}(x, y) = p(x, y)$ *and a polyradial body* L_2 *via* $\rho_{L_2}(x, y) = p(x, y)$.

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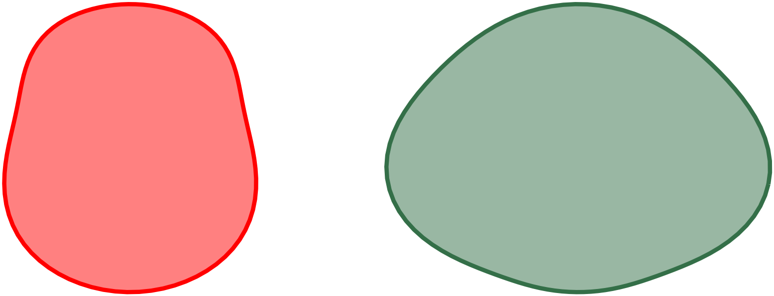

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Question.

- *How to construct a polystar approximation of E?*
- *Are there quantitative estimates of their accuracy?*
- *Are poly-star bodies good approximators?*

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$$h(x) = \int_S g(\langle x, y \rangle) f(y) d\mu(y)$$

where μ *is the* $(n-1)$*-dimensional volume measure.*

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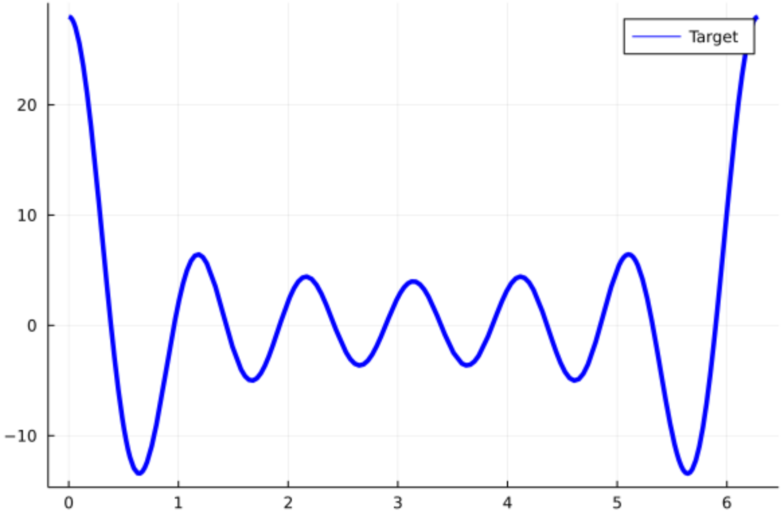

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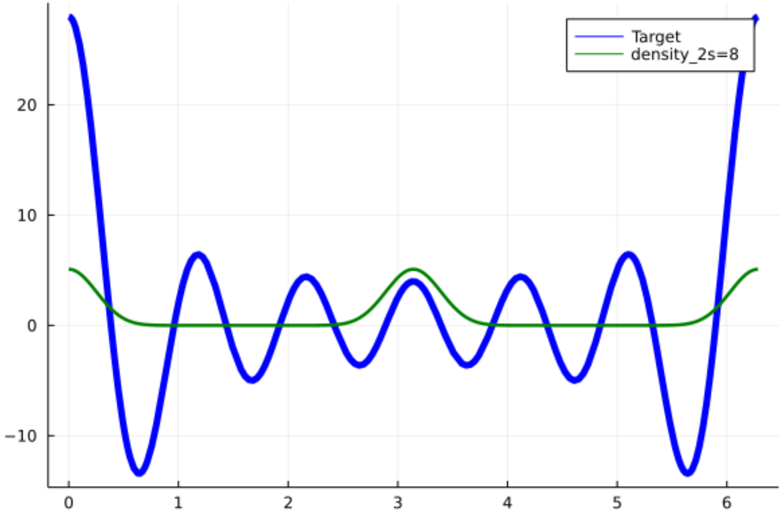

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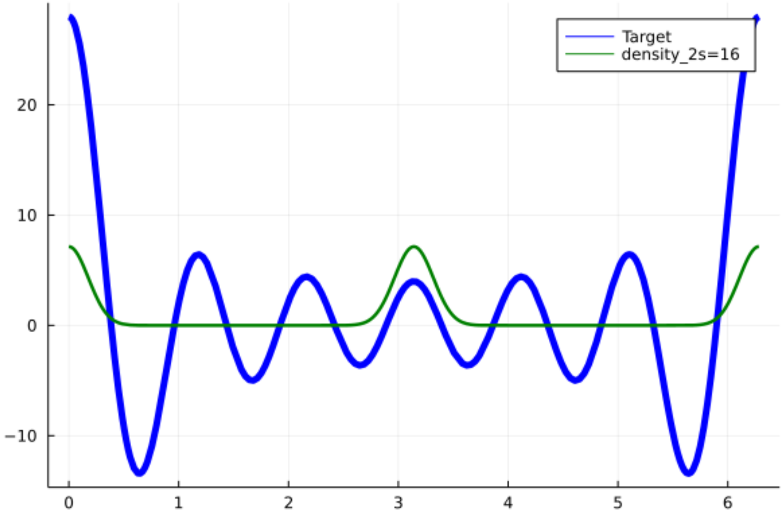

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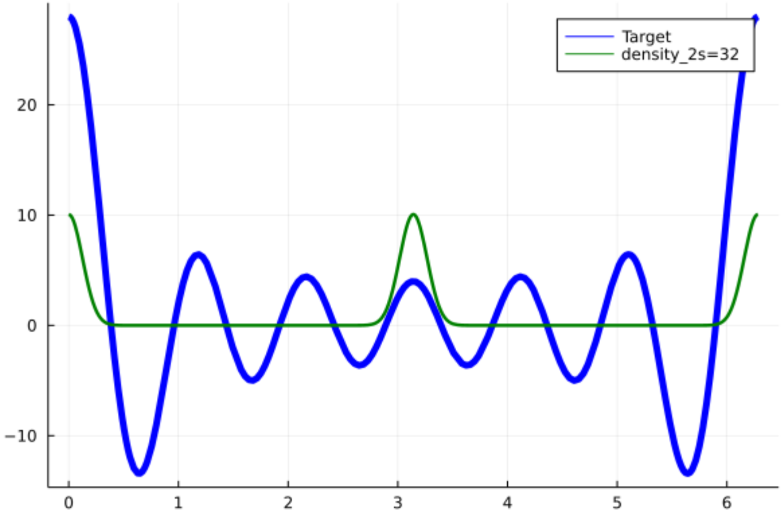

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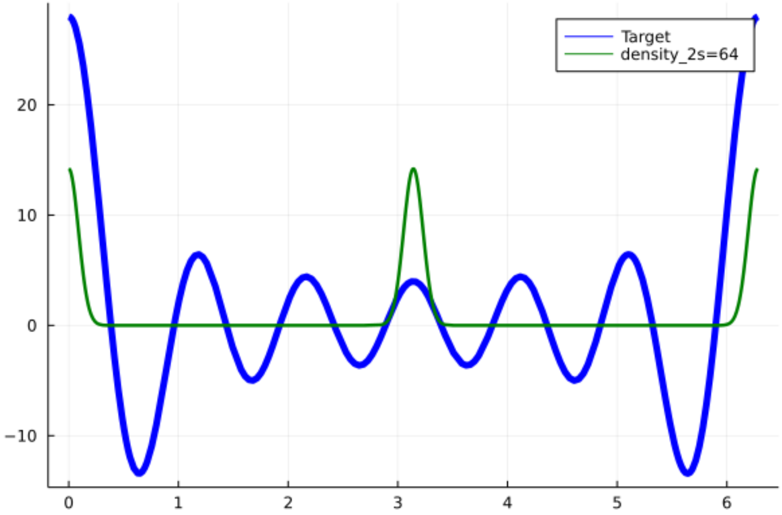

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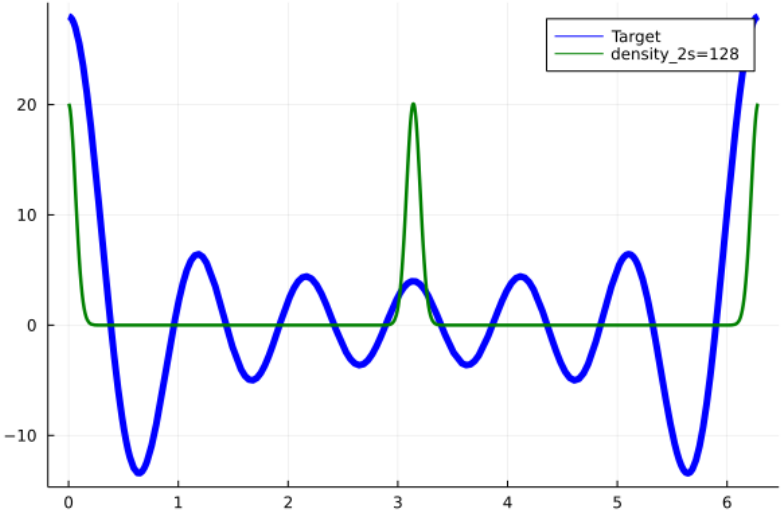

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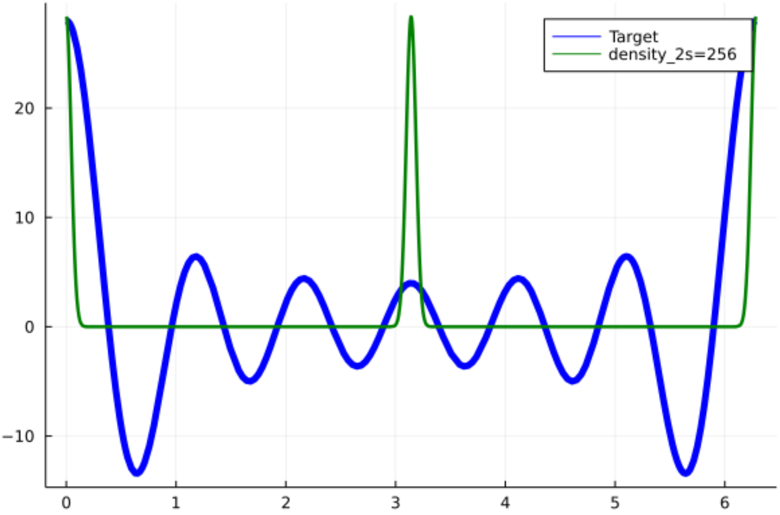

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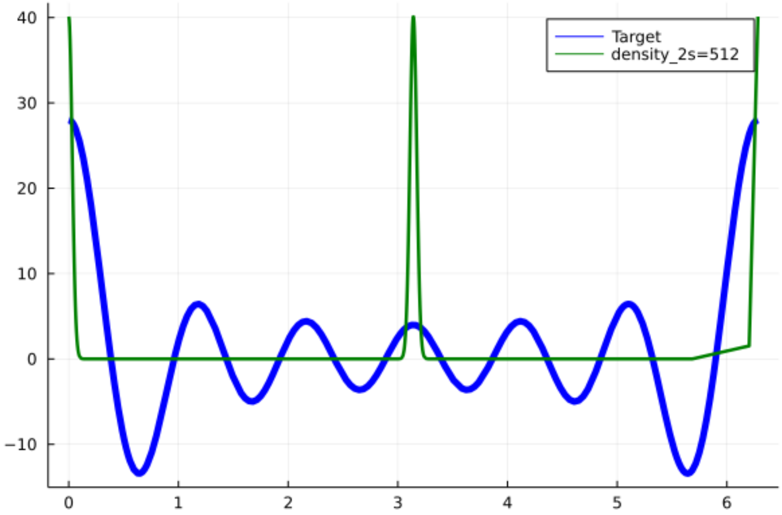

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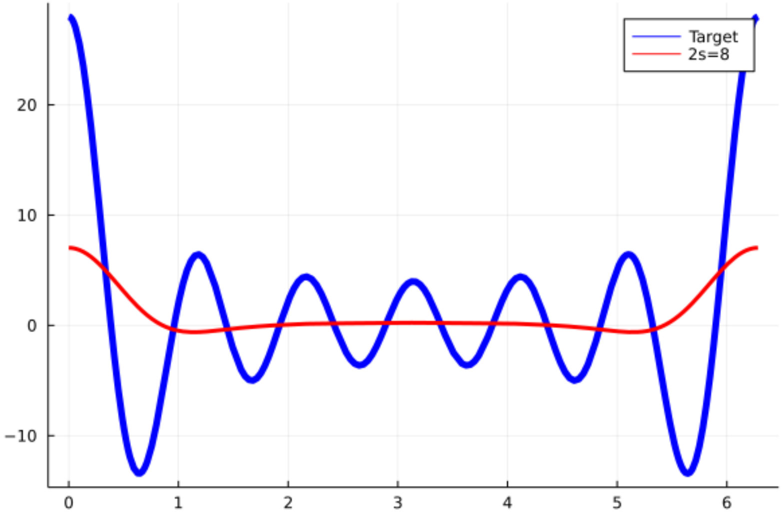

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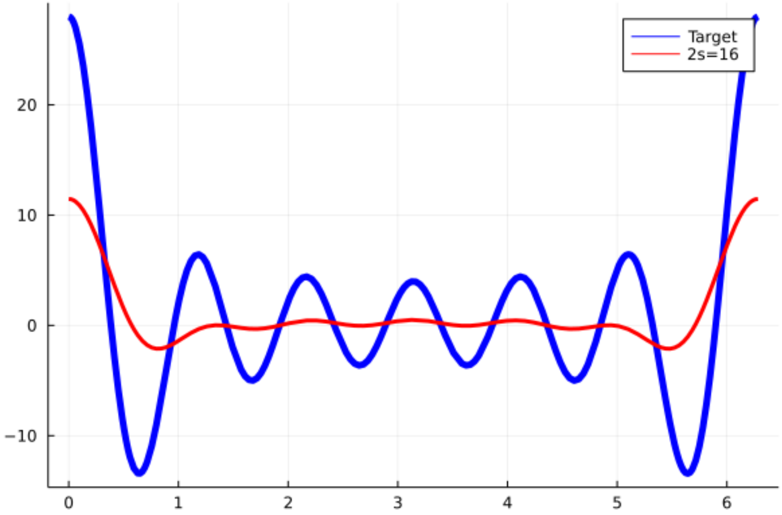

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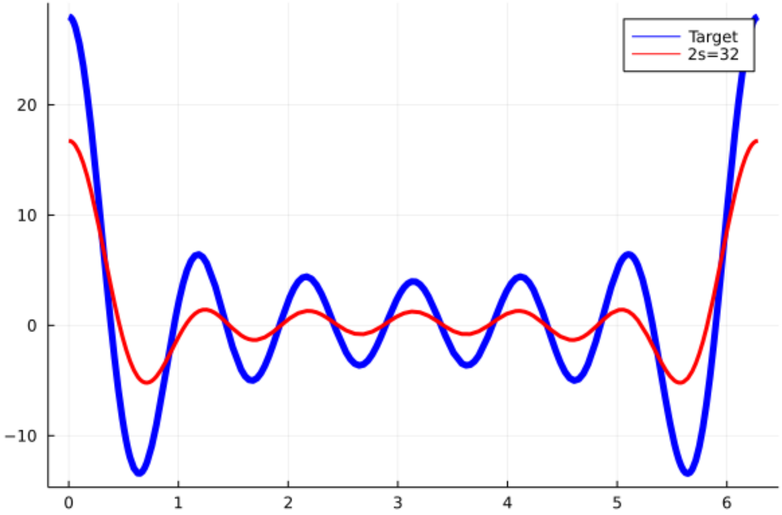

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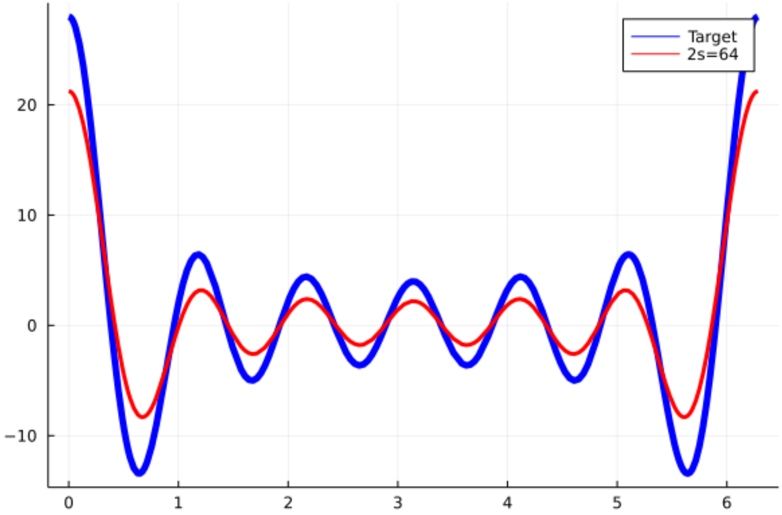

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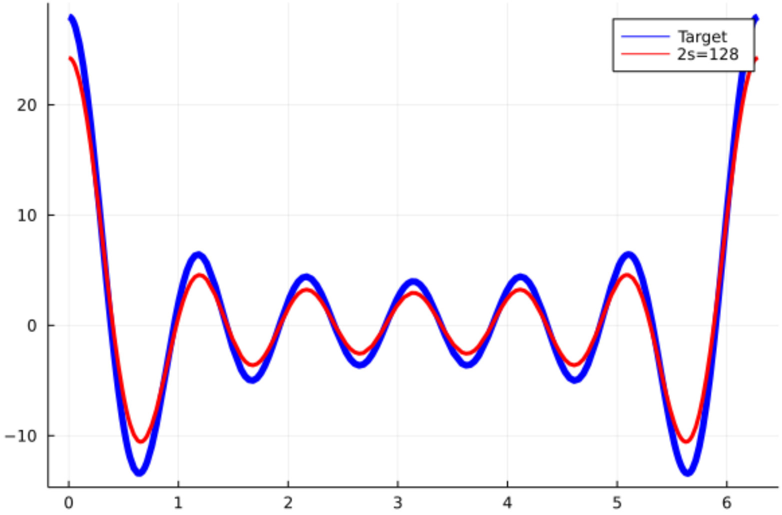

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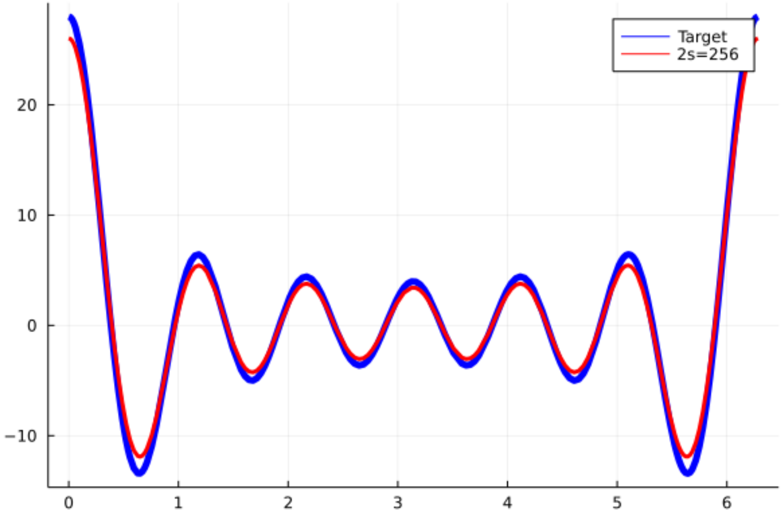

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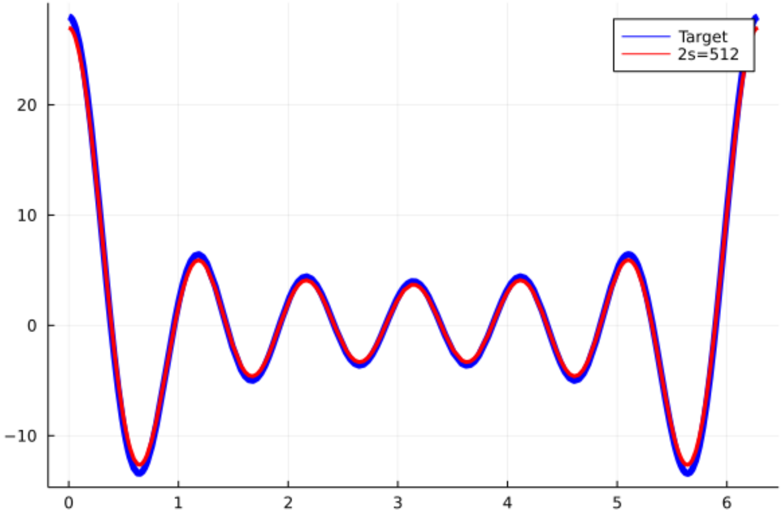

Constructing approximations

Theorem. (Miller, Meroni, -)

Let f be a Lipschitz function with Lipschitz constant κ on S^{n-1}. Then, there exists an explicit sequence of univariate nonnegative polynomials $\{g_d\}_d$ with $g_d : [-1, 1] \to \mathbb{R}$ of degree d such that for $d \to \infty$,

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We also proved that the polygauge approximating bodies of a convex body are **automatically convex**.

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Assume $n \geq 3$. Let f be a Lipschitz function with Lipschitz constant κ on S^{n-1}. Then, there exists a sequence of univariate nonnegative polynomials $\{g_d\}_d$ with $g_d : [-1, 1] \to \mathbb{R}$ of degree d such that for $d \to \infty$,

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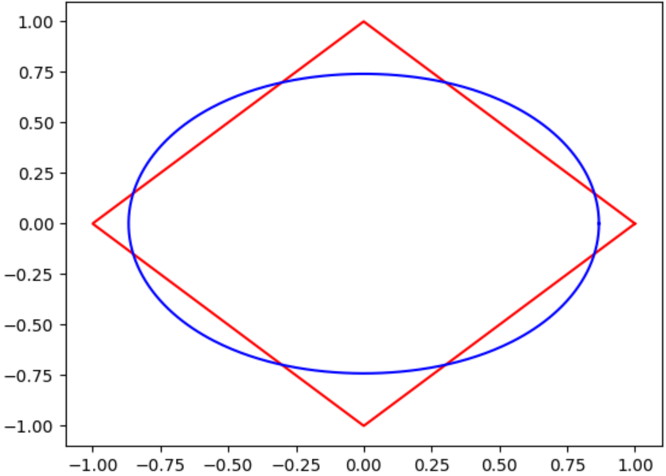
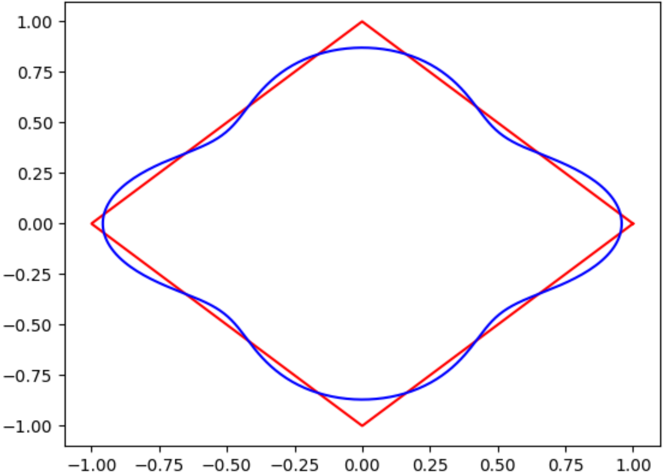
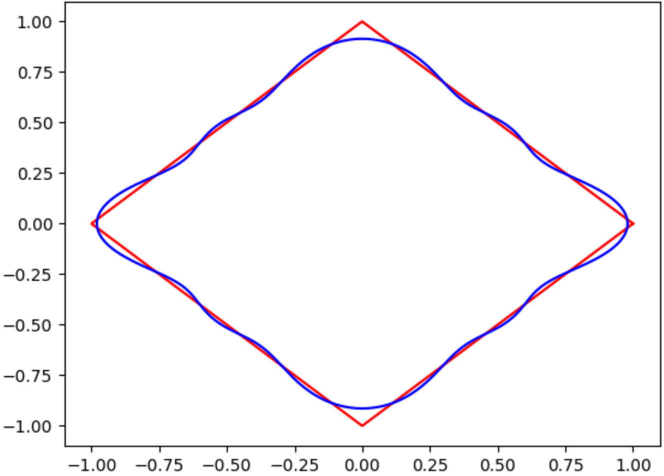
Example: degree 4



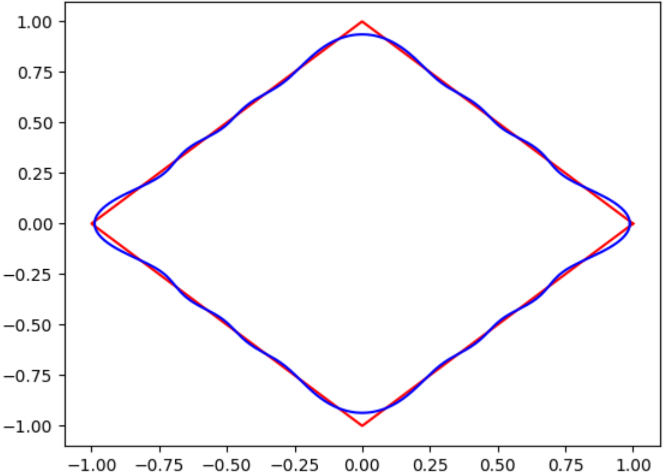
Example: degree 8



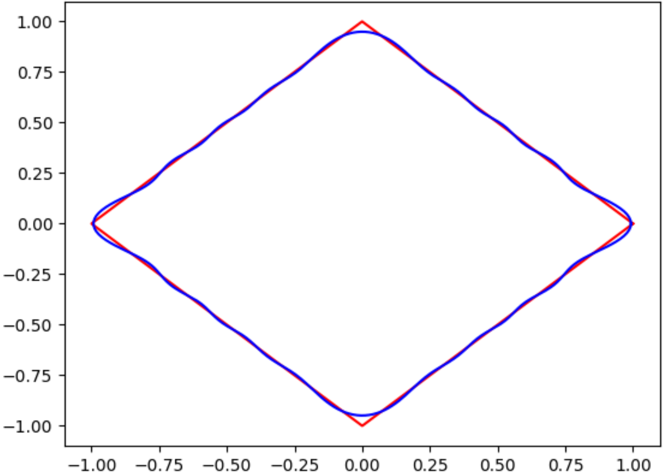
Example: degree 12



Example: degree 16



Example: degree 20



How to compute $T_g(f)$

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in other words $T_g : \mathbb{R}[S] \to \mathbb{R}[S]$ is a morphism of $SO(n)$ representations. This implies that all the maps Γ_g become simultaneously diagonal in **some natural basis**.

The ring $\mathbb{R}[S]$

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If $\mathcal{H}_d \subseteq \mathbb{R}[S]$ denotes the restrictions to the sphere of the homogeneous harmonic polynomials of degree d then

- $\mathcal{H}_d = \mathbb{R}[S]_{\leq d} \cap \mathbb{R}[S]^{\perp}_{\leq d-1}$ is an **irreducible** $SO(n)$ representation and
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In particular, every $f \in L^2(S^{n-1}, \mu)$ has a unique expression as a sum $f = \sum_{j=0}^{\infty} f_j$ with $f_j \in \mathcal{H}_j$, the **spherical harmonic decomposition** *of f.*

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Lemma. (Funk-Hecke)

Assume $g(t) = \sum_{j=0}^{d} \lambda_j^g \phi_j(t)$ is the unique expression of $g(t)$ as linear combination of (suitably normalized) Gegenbauer polynomials. If

$$f = f_0 + f_1 + \cdots + f_d + \ldots$$

is the spherical harmonic expansion of $f \in \mathbb{R}[S]$ then we have

$$T_g(f) = \lambda_0^g f_0 + \lambda_1^g f_1 + \lambda_2^g f_2 + \cdots + \lambda_d^g f_d.$$

Remark.

This implies that every equivariant map is essentially a univariate convolution (and in particular spherically symmetrical)

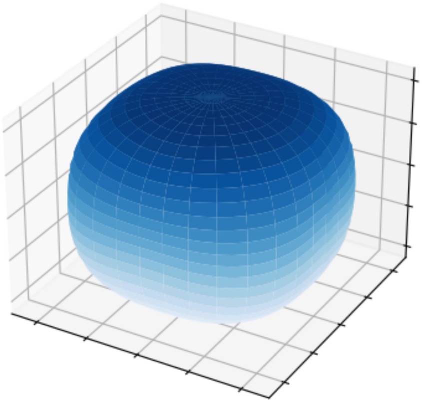
Strategy

All of the previous results lead to the following strategy:

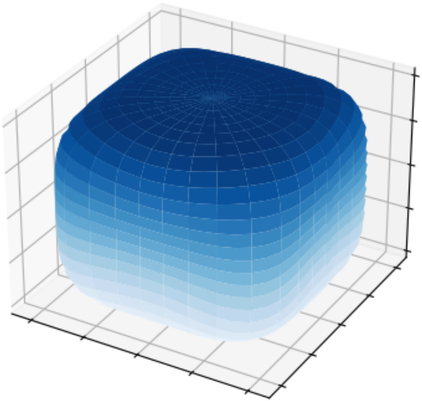
Given a black-box implementation of the radial/gauge fn. of E,

1. *Compute its harmonic components up to degree d.*
2. *Apply the Funk-Hecke formula with the special $g = u_d(t)$ to smoothen (mollify) the result.*
3. *Use the resulting star function to define a polyradial/polygauge body B which is a good uniform approximation for E.*
4. *The geometric invariants of B are often provably close to those of E and can often be computed exactly.*

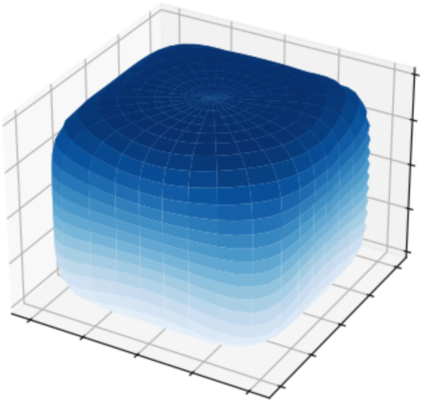
Example: degree 8



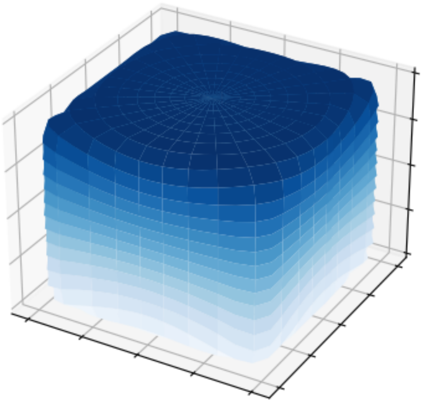
Example: degree 15



Example: degree 20



Example: degree 25



Application

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The intersection body IL of a starbody $L \subset \mathbb{R}^n$ is the starshaped set with radial function

$$\rho_{IL}(x) = \mathrm{Vol}(L \cap x^{\perp}) = \frac{1}{n-1} \int_{S^{n-1} \cap x^{\perp}} \rho_L(y)^{n-1} d\mu(y)$$

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$$\rho_{IL}(x) = \mathrm{Vol}(L \cap x^{\perp}) = \frac{1}{n-1} \int_{S^{n-1} \cap x^{\perp}} \rho_L(y)^{n-1} d\mu(y)$$

This quantity equals $\mathcal{R}\left(\frac{1}{n-1}\rho_L^{n-1}\right)(x)$, where $\mathcal{R}$ denotes the spherical radon transform of any function in $L^2(S^{n-1}, \mu)$.

Question.

What is the Intersection Body of the cube $[-1,1]^3$?

Example: IB of the cube $[-1, 1]^3$

IB link

Approximation Quality

It is natural to ask whether the proposed approximation of polystar bodies by polyradial bodies is efficient and whether there are much better choices than polynomials.

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Definition.

If $A \subseteq C(S^{n-1})$ is a set and N is an integer the Kolmogorov width of A is defined as

$$\mathcal{W}_N(A) := \inf_{W \subseteq C(S)} \left(\sup_{a \in A} d(a, V) \right)$$

where the infimum runs over all linear subspaces $V \subseteq C(S)$ of dimension at most N.

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In words, the Kolmogorv N-width measures the smallest (best possible) worst-case uniform approximation error among all subspaces V of dimension N when we want to approximate the functions in A uniformly.

Approximation quality

For any positive constant $\kappa > 0$, let $\Lambda(\kappa)$ be the set of radial functions of star-bodies with Lipschitz constant at most κ.

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We conclude that polyradial bodies are asymptotically optimal approximators.

Approximation quality

Theorem. (Meroni, Miller, -)

Given $\kappa > 0$, for all $N \in \mathbb{Z}$ large enough and any positive real β such that $N\beta^{n+1} < 1$, the inequality $\mathcal{W}_N(\Lambda(\kappa)) \geq \frac{\kappa\beta}{2}$ holds. In particular, there exists a positive constant C_0 such that for all sufficiently large d there exists a starbody L with $\rho_L \in \Lambda(\kappa)$ such that

$$\|\rho_L - p\|_\infty \geq C_0 \frac{\kappa}{d}$$

for all polynomials p of degree d on S.

We conclude that polyradial bodies are asymptotically optimal approximators.

Remark.

The proof extends an argument due to G.G. Lorentz [1960] from Lipschitz functions in $\mathbb{R}^n$ to star-bodies.